SEMI-PLANAR PERFORMANCE ANALYSIS AND *KISSING CIRCLE* SENSITIVITY

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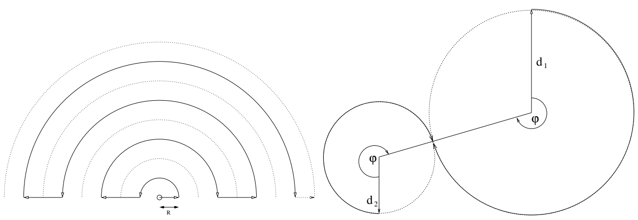
ABSTRACT. This paper looks at the problem of searching the plane for an *object* with multiple agents, in co-ordinated search. Also the problem of how to deal with noise within the context of Fakete's [3] Kissing Circles algorithm is addressed and a method of coping is conjectured.

The relative performance of a family of redezvous search algorithms for mulit-agent search in the plane. Some very strict assumptions have been made to idealize the circumstance in order to simplify the analysis.



1. THE PROBLEMS

The problems that we will consider here relate to multi-agent rendezvous. As a basis we are attempting to extend the work that has been done analyzing optimal searching methods for single agents under various restictions (see [4]) in concert with rendezvous strategies to effectively co-ordinate exploration. These considerations hopefully show the motivation for the strict assumptions that this beginning work makes. Future work will be proposed in the paper that will relax these restrictions to deal with more realistic circumstances.

Two friends have lost their car keys. They remember putting them down somewhere in the plane, but niether has any idea where. So we have uniformly distributed knowledge of the solution space, i.e. none. How should they go about finding them? Bear in mind that the friends are lazy, and the plane is a big place. What should they do? If they heed the advice of [4] they would understand that spiral search will be at least as good as any method for getting around. Since they feel they should split up to find their keys faster, then they probably want to perform *bow-tie* search on their respective search regions since this, too, is at least as good as anything else they could think of.

Our friends are concerned though that when one of them find the keys, how will they tell the other that it is time to go? This seems like a problem since they both walk the same speed, and it might take too long to catch up to the other person after finding the keys. In addition, our friends can't bear to yell out, they have head-aches [perhaps this is why they have lost their keys!]. By whispering to one another they restrict their range of communication to almost zero. So they agree to [try to] meet every so often at the origin of there search. But how often is good enough for these lazy friends? This is what we will try to answer for them.

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Another problem that Fakete [3] tries to help our friends with is how to meet up with each other when they are lost in the bush. With their head-aches all cleared up, and being outside, our friends are free to call out to one another and by doing so can judge how far away they are from each other. Though the bush is thick and they can't tell which direction to walk in to find each other. Fortuneately they both have compasses, but one friend has hurt his leg and can't walk as fast as usual, but this shouldn't matter. A problem has arisen though. During the fall, when she injured her leg she also damaged the compass. So although it still works, albeit inaccurately, can they still find each other? Perhaps not using the same method they first thought, but by using a very similar one they can maximize there chances of reuniting.

2. Semi-Planar Rendezvous Search Recast as a Synchronization Problem in Parallel Breadth-First Search

For interest sake this observation has been included to illustrate the generality of the rendezvous search problem. Trying to perform redezvous search has some very key features that are pervasive in computer science. The problems of task load balancing, especially on multi-processor systems, and process synchronization in parallel BFS seem to share many of the complexity problems that we have to deal with here.

The trade-off between the expected gain from continued search must be balanced against the effort taken to synchronize the global process, to minimize the amount of wasted search when a solution has been found by one of the processes [agents].

The similarity between the rendezvous search and parallel BFS is as follows. Both have the property that the next search region is polynomially larger than the current one. Having finished a level of search the alternatives are to,

1. Continue to search the next region for solution.
2. Communicate with the other agents/processes to see if they have found a solution yet.

The question is, once we have an effiecient method of search for the single agents, how do we balance off the relationship between search and co-ordination. The ratio to optimize [*maximize*], in this case, can be seen as,

$$\frac{Ex(\texttt{gain from search})}{Cost(\texttt{rendezvous})} \tag{1}$$

the expected gain versus the cost of co-ordination. This is the difficulty in any parallel system, to minimize the overhead cost of cooridination. Why co-ordinate at all? Co-ordination reduces the amount of wasted search after an agent discovers a solution. It is the intention to minimize this waste, in the interest of laziness or rather efficiency.

3. Rendezvous Search in the Plane

We consider the case of 2 unit speed agents searching the plane for a single target. The agents can sense a region in the plane in a disk about their position of radius, r. We will assume that both agents adopt the same search strategy, so that without loss of generality, we can perform analysis on the half-plane and a single agent[1].

[1] The *assymetric* case where each agent may adopt a different strategy is a future expansion of this analysis, that would have to include the probability of failed rendezvous attempts in the cost function

Following the advice of [4] we will adopt the approprite model of spiral search as the exploration strategy. In this circumstance, spiral search simplifies to walking concentric circles of odd radius, so that the boundary of the sensed regions just touch. This leaves a measure zero set ($\mathcal{R}^2/\mathcal{R}^2$) of the plane unsearched so that we are guaranteed to find the target if it lies within the searched region. See figure 1 for a description of the search pattern.

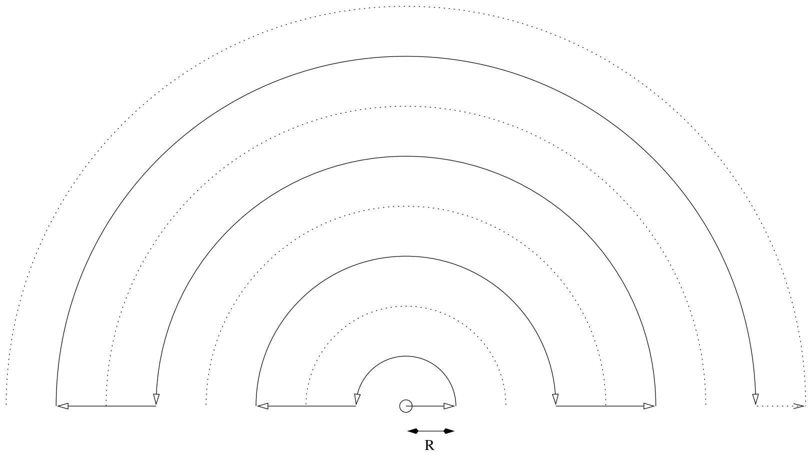

FIGURE 1. The path traced by σ_∞ of concentric hemi-circles.

The family of rendezvous strategies that we will consider can be described as follows. Every time our agent arrives back at the border of the half plane she has a choice to move back to the origin to meet with the other agent, or continue to search the next level out of her region. We will evaluate and analyze the cost of the alternate behaviors and attempt to categorize the strategy family. Let us describe the strategies in the following manner, let $\sigma_i \in S$ be the strategy that rendezvous is attempted upon every i-th arrival at the search boundary. Let $\mathcal{Z}$ be the positive integers then redezvous is attempted at border encouters from the set, $\{\mathcal{Z}/\mathcal{Z}_i\}$, for strategy σ_i. Define σ_∞ to be the strategy of spiral search without rendezvous attempts (since $\{\mathcal{Z}/\mathcal{Z}_\infty\} = \emptyset$). Consider the distance that is traversed during this search,

$$
\begin{aligned}
D(\sigma_\infty) &= r + \pi r + 2r + 3\pi r + 2r + 5\pi r + 2r + 7\pi r + \cdots \\
&= \sum_{i=1}^{\infty} 2r + (2i-1)\pi r
\end{aligned}
\tag{2}
$$

In fact we can easily go ahead and categorize the distance traversed by each strategy in the family, by applying similar analysis. Note that the sum is chosen to reflect the distance travelled between rendezvous attempts, breaking the infinite sum in this way will be useful when analyzing the strategies according to rendezvous attempts and we can safely take the sum term-wise due to the compactness of the plane. Yeilding the sequence,

$$
\begin{aligned}
D(\sigma_1) &= r + \pi r + 4r + 3\pi r + 8r + 5\pi r + 12r + 7\pi r + 16r + \cdots \\
&= \sum_{i=1}^{n} (4i-2)r + (2i-1)\pi r \\
D(\sigma_2) &= r + \pi r + 2r + 3\pi r + 8r + 5\pi r + 2r + 7\pi r + 16r + \cdots \\
&= \sum_{i=1}^{n} (8i-2)r + (8i-4)\pi r \\
D(\sigma_3) &= r + \pi r + 2r + 3\pi r + 2r + 5\pi r + 12r + 7\pi r + 2r + \cdots \\
&= \sum_{i=1}^{n} (12i-2)r + (18i-9)\pi r \\
&\vdots
\end{aligned} \tag{3}
$$

$$
D(\sigma_j) = \sum_{i=1}^{n} (4ji-2)r + (2j^2 i - j^2)\pi r \tag{4}
$$

taking the sum, of course as $n \to \infty$. But splitting the sums up in this fashion allows us to see the pattern that evolves by the members on each of the respective rendezvous attempts. The pattern that evolves is intuitive enough since for each member the distance travelled between rendezvous iterations consecutive odd traversals of hemi-circles, followed by the trip to the origin. This acounts for the terms in (4), the πr term for the exploration trips, and the r term accounting for the rendezvous trips.

Now we can characterize the distance that is wasted, D_w, perfoming rendezvous by taking the difference of (4) and (2) resulting in the extra distance travelled at each iteration.

$$
\begin{aligned}
D_w(\sigma_j - \sigma_\infty) &= \sum_{i=1}^{n} (4ji-2)r + (2j^2 i - j^2)\pi r - \sum_{i=1}^{n} 2r + (2i-1)\pi r \\
&= \sum_{i=1}^{n} 4jir, \qquad \forall j, \texttt{when } n \to \infty
\end{aligned} \tag{5}
$$

How then does this figure into our competitive ratio (1)? If this is the accumulated distance that we waste at each iteration then our opportunity cost is searching this far into the next level of the exploration. So what is our expected gain from further search? It is the area, A, that we will see on the next iteration of the search strategy. This is easy enough to figure out, we have already seen this as the area, πr, term in (3).

$$
\begin{aligned}
A(\sigma_1) &= r\left((\pi) + (3\pi) + (5\pi) + (7\pi) + \cdots\right) \\
&= \sum_{i=1}^{n} (2i-1)\pi r \\
A(\sigma_2) &= r\left((\pi + 3\pi) + (5\pi + 7\pi) + \cdots\right) \\
&= \sum_{i=1}^{n} (8i-4)\pi r \\
A(\sigma_3) &= r\left((\pi + 3\pi + 5\pi) + (7\pi + 9\pi + 11\pi) + \cdots\right) \\
&= \sum_{i=1}^{n} (18i-9)\pi r \\
&\vdots
\end{aligned} \tag{6}
$$

$$
A(\sigma_j) = \sum_{i=1}^{n} (2j^2 i - j^2)\pi r \tag{7}
$$

Then from (1) we see that what we expect to gain is area of the next iteration of search for our strategy, σ_j. This is in inverse proportion to the accumulated distance that we have walked for rendezvous purposes. That proportion is what we need to understand.

$$
\begin{aligned}
\frac{Ex(\texttt{gain from search})}{Cost(\texttt{rendezvous})} &= \frac{Ex(\texttt{Area next Level})}{D_w(\sigma_j - \sigma_\infty)} \\
&= \frac{(2j^2 n - j^2)\pi}{\sum_{i=1}^{n-1} 4ji} \qquad \texttt{from (7),(5)} \\
&= \frac{(2j^2 n - j^2)\pi}{\frac{4j(n^2-n)}{2}} \\
&= \frac{(2j^2 n - j^2)\pi}{2jn^2 - 2jn}
\end{aligned} \tag{8}
$$

The function is plotted in figure 2 and is seen to have quadratic cost in terms of the strategy chosen. It makes sense to have strategies of $j < 1$, which would correspond to not exploring the entire semicircle before returning to the rendezvous origin, perhaps in circumstance where the sensing region was very large relative to the speed moved, $v \ll R$.

4. Deviation on Kissing Circles

So now our friends have found their keys, and informed each other of that fact [unless of course they chose σ_∞ in which case they can't find each other now!]. So it is time to head for the country for a walk in the bush to clear their heads, after that tiresome search. During their walk they become separated and although they can ascertain the distance between them, they can't find each other. Fortuitously, they both were brushing up on their geometry during the ride and are familiar with the method supplied to them by Fakete([3]).

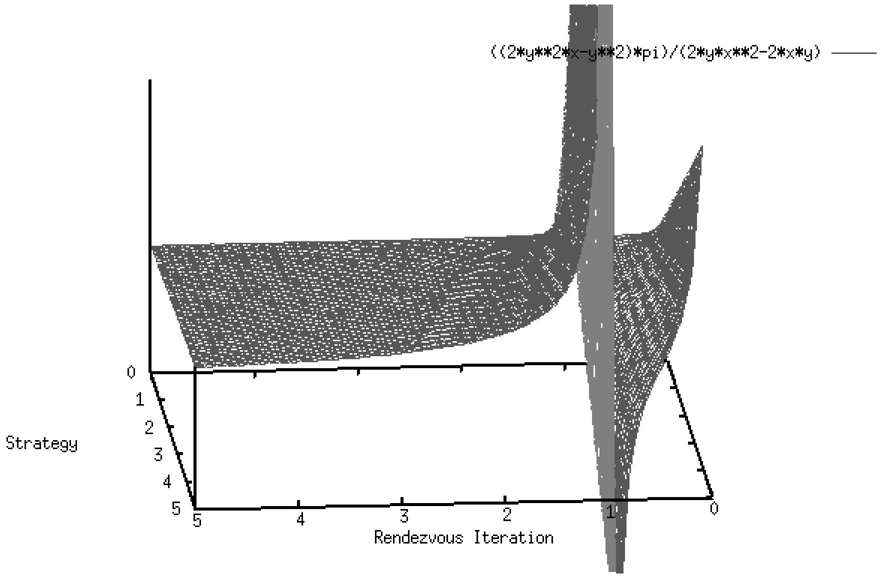

FIGURE 2. The Competitive Ratio (1) of exploration versus rendezvous overhead

By following the solid lines in Figure 3 they are assured of doing no better at finder one another with their eyes closed, so to speak. This is assuming that their compasses are perfectly calibrated and the Earth being flat [a fact which we know *not* to be true, see [5]]. One of the friends initially

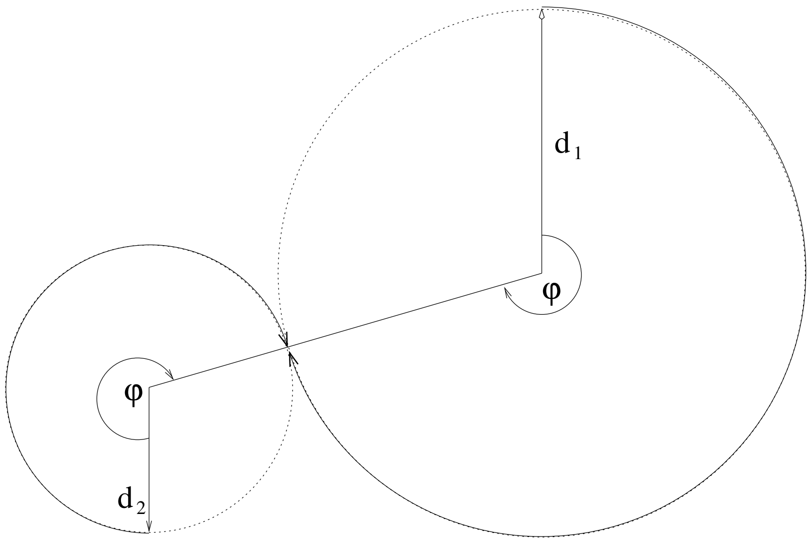

FIGURE 3. The Kissing Circles algorithm, with $r = 0$

walks south, and the other north, then they follow a circle about their starting positions until they bump into one another [See Fakete [3] for a full explaination].

By introducing disparity in the compass readings this algorithm will not succeed. Provided our agents open their eyes and so can see about themselves a radius, $r > 0$, we conjecture that the following modification to the Kissing Circle algorithm will maximize their chances of reuniting. It goes as follows.

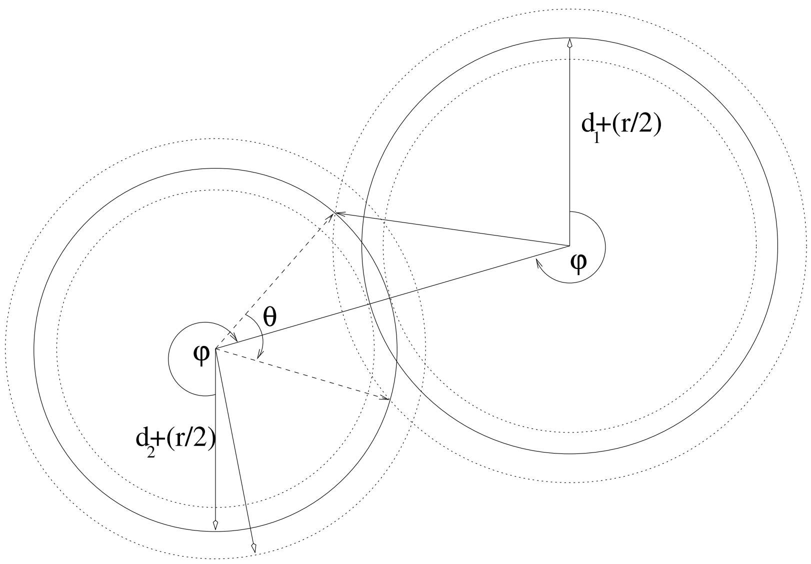

FIGURE 4. The modified Kissing Circles algorithm, with $r > 0$

Again, one walks south and the other north, according to their personal compasses. Now, however, it is conjectured that if they walk the same distance they walked before plus half the distance they can sense, $D_i = \frac{dv_i}{v_1+v_2} + \frac{r}{2}$, $\forall i \in \{1, 2\}$, and follow the circle etched out by that radius then they will find each other somewhere in the arc spanned by θ in figure 4. We believe this method maximizes the discovery arc-length.

5. FUTURE INVESTIGATIONS

In terms of where this work should lead, there are many things that have to happen before any of this is ready for the "real world". Noise is always an issue when real sensors are used movement is imprecise. Tolerance for noise will have to be handled everywhere. Presumeably that is the whole motivation for the section on the deviated Kissing Circles algorithm.

Perhaps rendezvous at the origin isn't the most appropriate model to work with. If, upon discovery of the target, one agent were to try to track down the other, using knowledge of the strategy being employed, then some of the overhead in rendezvous could be reduced in favour of a greater expected exploration payoff.

Asymmetric strategies should also be considered. Perhaps of the flavour of introducing a probability of attempting a rendezvous after some given iteration or just continuing to search with a complimentary probability. Thereby introducing the possibility of failed rendezvous attempts. An effect that will have to be tolerated any way under noisy circumstances.

Clearly the conjecture made in the last section needs to be affirmed. Hopefully though, this work is just the core for more to come.

Excelsior.

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