UPPER BOUND CONDITIONING AS A PERFORMANCE INDEX FOR MANIPULATOR MOTION PLANNING

René V. Mayorga - Luz Abril Torres

Department of Systems Design Engineering
University of Waterloo
Waterloo, Ontario, Canada.

Abstract

In this article a simple performance index for proper manipulator motion planning is presented. This index is derived by establishing a simple upper bound for a *standard* condition number of the Jacobian matrix. Here the proposed index behaviour is analyzed/tested on a planar redundant manipulator executing tasks under various scenarios.

1 Introduction

In the past decade several studies on (redundant) manipulators design have been performed for the kinematic evaluation of some designs [2], [4], and [9]. The main objective of these studies is to develop a dexterity measure (with respect to distance to singularities) by establishing a performance index: derived from the singular value decomposition of the Jacobian matrix [2], [4]; or form the square root of the determinant of the Jacobian matrix by its transpose [9]. These approaches for some particular cases can provide local performance (on neighbor or distant singularities regions) information [5]. However, they are not suitable for a simple global design analysis, neither a useful for the comparison of different designs [6]. Also, in the general case they can not be utilized to measure manipulator performance invariant to frame reference selection and/or scaling [6]. Furthermore, most of these studies have been focused on a kinematic design optimization, neglecting the analysis/test behaviour of the performance indices in the entire manipulator task/performance space.

Recently in [6] a simple kinematic criterion has been developed for simple manipulator design optimization (with respect to Jacobian matrix well-conditioning). Here, as in [6] an upper bound for the *standard* condition number of the Jacobian matrix at a given joint configuration $\theta(t_i)$ at instant t_i is established. It can be easily shown [6] that this upper bound constitutes a sufficiency condition for the preservation of the rank of the Jacobian matrix at $\theta(t_{i+1})$; and that also constitutes an upper bound for a *standard* condition number of the Jacobian matrix at $\theta(t_{i+1})$.

It also has been demonstrated in [6] that the established bound can serve as a performance criterion. From its explicit form a simple constrained optimization problem can be easily formulated to obtain some optimal manipulator parameters, as well as the best posture. Notice that, unlike other performance indices that require extensive simulations, the proposed simple criterion [6] can be easily used for general cases.

Here, the proposed criteria [6] behaviour is analyzed/tested, and compared with other criteria, on the motion planning of a planar redundant manipulator performing several tasks in the entire space.

2 Kinematic performance indices

It is a well known fact that the generation of smooth and bounded joint velocities greatly depends on the rank preservation and conditioning of the Jacobian matrix $J(\theta)\epsilon\Re^{(mxn)}$ [7]. In general, current indices/measures of manipulator performance conceptually represent a distance to singular configurations [9], [3]; or a well conditioned Jacobian matrix [5], [3]. They are based on the singular value decomposition of the Jacobian matrix [9]; that is

$$J(\theta) = U(\theta)\Sigma(\theta)V^T(\theta), \tag{1}$$

where $U\epsilon\Re^{(mxm)}$, and $V\epsilon\Re^{(nxn)}$, are orthogonal matrices and $\Sigma\epsilon\Re^{(mxn)}$, contains the singular values $\sigma_1(\theta) \geq \sigma_2(\theta) \geq ... \geq \sigma_m(\theta)$ as its jj element; $j = 1, 2, ..., m$; and all the other elements equal to zero. By considering these values, the following measures can be defined:

- (a) based on a determinant (T. Yoshikawa [9]),

$$p(\theta) = (detJ(\theta)J^T(\theta))^{1/2} = \sigma_1(\theta)\sigma_2(\theta)...\sigma_m(\theta); \tag{2}$$

- (b) base on the smallest singular value (C.A. Klein [3]),

$$p(\theta) = \sigma_m(\theta); \tag{3}$$

- (c) based on the condition number (C.A. Klein [3]);

$$p(\theta) = \sigma_1(\theta)/\sigma_m(\theta); \tag{4}$$

Notice that these measures depend on the singular values, which are difficult to express explicitly in terms of the joint configuration. Consequently, they can be used mainly for particular cases. Although they are useful to determine a best configuration/posture and an appropriate operation subspace of a particular manipulator [2]; their application for general manipulators can be very complex [6]. A somewhat simpler criterion, based on an isotropy criterion, that can be used for general manipulators has been recently presented in [1]. However, in general, all these measures are not invariant to frame reference selection and/or scaling [6].

The indices provided by Eqs.(3), and (4), [2],[3], [4] have also been utilized for the motion planning of a particular manipulator [5]. However, their implementation for the real-time motion planning of general manipulators is still cumbersome.

Recently in [6] a simple kinematic criterion has been developed for the kinematic design optimization, with respect to a *standard condition number*, of general manipulators.

3 Proposed Kinematic Criterion

First, consider $m \leq n$ and let the (mxn) matrix $J(\theta(t))$ be the Jacobian matrix at any $t\epsilon[t_0, t_f]$. Also, unless otherwise specified, let here $\| . \|$ stand for the $p-$norm $\| . \|_p$, $p = 1, 2, \infty$.

Now let $\dot{x} = [v, \gamma]^T \epsilon SE(3)$ [6]; that is, the vectors v and γ correspond to end-effector positions and orientations respectively. As in [6], let's define a standardized norm in $SE(3)$ as:

$$\| \dot{x} \|_s \equiv (\dot{x}^T Z \dot{x})^{1/2} = \| Z^{1/2} \dot{x} \|_2; \tag{5}$$

where the positive definite symmetric matrix Z acts as a metric to allow invariance to frame selection and/or scaling [6]. Now, also without losing generality consider $\dot{\theta} = [r, p]$ belonging to the *joint space* composed of revolute and prismatic joints; that is, the vectors r and p correspond to *revolute* and *prismatic* joints respectively. Also as in [6], let's define a standardized norm in the *joint space* as:

$$\| \dot{\theta} \|_s \equiv (\dot{\theta}^T W \dot{\theta})^{1/2} = \| W^{1/2} \dot{\theta} \|_2; \tag{6}$$

where the positive definite symmetric matrix W acts as a metric to homogenize dimensions [6].

From the previous definitions a standardized condition number in $SE(3)$ can be established. As shown next, this condition number serves to develop a kinematic criterion to measure manipulator performance [6].

Proposition 1 *Let $\tilde{\sigma}_1 \equiv \sigma_1 Z^{1/2} J W^{-(1/2)}$, and $\tilde{\sigma}_m \equiv \sigma_m Z^{1/2} J W^{-(1/2)}$, be the largest and smallest singular values of the matrix $[Z^{1/2} J W-(1/2)]$ respectively. Then, a standard condition number can be given by:*

$$k_s \equiv \| J \|_s \| J^+ \|_s = \tilde{\sigma}_1 / \tilde{\sigma}_m. \tag{7}$$

Now, let's define

$$\bar{J}(\theta(t_i), \theta(t_i), \ddot{\theta}(t_i)) \equiv \dot{J}(\theta(t_i), \dot{\theta}(t_i))\Delta t +$$

$$(1/2)\ddot{J}(\theta(t_i), \dot{\theta}(t_i), \ddot{\theta}(t_i))\Delta t^2 + O(\Delta t^3); \tag{8}$$

where

$$\bar{J}(\theta(t_i), \theta(t_i)) = d[J(\theta(t_i))]/dt; \tag{9}$$

and $O(\Delta t^3)$ is an (mxn) matrix of third order terms. Notice that for $t_i \epsilon [t_0, t_f]$

$$J(\theta(t_{i+1})) = J(\theta(t_i)) + \bar{J}(\theta(t_i), \dot{\theta}(t_i), \ddot{\theta}(t_i)). \tag{10}$$

Now for convenience, let's drop the index t_i in the subsequent expressions. It can be easily shown that an upper bound for $\bar{J}(\theta(t_i), \dot{\theta}(t_i), \ddot{\theta}(t_i))$ is given by:

$$\| \bar{J}(\theta, \dot{\theta}, \ddot{\theta}) \|_s \leq l(\alpha + \beta) \tag{11}$$

where

$$\alpha \equiv \Delta t \sum_{j=1}^{n} \| \hat{J}_j \|_s + \Delta t (l_a / 2l) \|_s \tag{12}$$

$$\beta \equiv (l \Delta t^2) \sum_{j=1}^{n} \sum_{k=1}^{n} \| \breve{J}_{jk} \|_s + \| O(\Delta t^3) \|_s . \tag{13}$$

where $\hat{J}_j \equiv \hat{J}_j(\theta) \equiv \partial J(\theta)/\partial \theta_j$; $\breve{J}_{jk} \equiv \partial \hat{J}_j / \partial \theta_k$; $j, k = 1, 2, ..., n$; and $l = max \mid \dot{\theta}_j \mid$; $j = 1, 2, ..., n$; and $l_a = max \mid \ddot{\theta}_k \mid$; $k = 1, 2, ..., n$. Notice that l can be considered arbitrarily, and that $l \Delta t \approx \Delta \theta_{max} \equiv max \mid \Delta \theta_j \mid$; $j = 1, 2, ..., n$. Also notice that the α, and β are dimensionless.

Now, let's consider an upper bound (in terms of α, and β) for the condition number of the Jacobian matrix at a given joint configuration $\theta(t_i)$ as follows:

$$\mu(\theta(t_i)) \equiv \| J(\theta(t_i)) \|_s / l(\alpha + \beta) \tag{14}$$

The next Theorem states that such an upper bound constitutes a sufficiency condition for the preservation of the rank of the Jacobian matrix at $\theta(t_{i+1})$ [6].

Theorem 1 *Let $m \leq n$, $p = 2$; α, and β as in Eqs.(12), and (13); and μ as in Eq.(14). Also, suppose that at $t_i \epsilon [t_0, t_f]$, $Rank J(\theta(t_i)) = m$. If the condition number k_s is bounded as follows*

$$k_s(\theta(t_i)) \leq \mu(\theta(t_i)) \tag{15}$$

Then, $RankJ(\theta(t_{i+1})) = m.$

Moreover, the following Theorem can also be easily shown [6].

Theorem 2 *Let the Assumptions of the Proposition 1, and the Ineq.(15) be satisfied. Then, the condition number of the matrix* $J(\theta(t))$ *at* t_{i+1} *can be expressed as*

$$k_s(t_{i+1} \leq 2\mu(t_i)\tilde{\sigma}_1(t_{i+1}/[\tilde{\sigma}_m(t_i) - [l(\alpha+\beta)]]. \qquad (16)$$

In [6] the upper bound is utilized as a criteria for manipulator design optimization, and to indicate a best posture. From Eq.(16) it can be observed that this upper bound can have a significant effect on the condition number (and consequently on manipulator motion planning) in the entire space. The criteria behaviour for manipulator motion planning under various Task scenarios is analyzed in the next section.

4 Criteria behaviour

Here, the behaviour of the proposed criterion is analyzed/tested on the motions of a planar redundant manipulator (resulting from the forward or inverse kinematic equations) performing several tasks in the entire space. The behaviour and results obtained are compared with the ones obtained using other criteria/measures [1], [2], [9]. As previously mentioned, these criteria have been devised to determine a measure of ill-conditioning of the Jacobian matrix, or a distance to its singular configurations. The singular configurations are undesirable joint configurations characterized by the Jacobian matrix losing its rank, and inducing extremely large joint velocities for small end-effector changes. Consequently, a throughly simulation of manipulator motions and analysis of the performance criteria should focus on the motions *near* and *far* from those undesirable singular configurations.

Notice that from Eq.(4) it is relatively easy to show that an upper bound for the condition number is given by:

$$k_2 \equiv \sigma_1(\theta)/\sigma_m(\theta) \leq \| J(\theta) \|_F^m / \delta^{1/2}(\theta); \qquad (17)$$

where, $\delta(\theta) = det[J(\theta)J^T(\theta)]$. Also, it is easy to show that (for $m \leq n$) the isotropy condition implies:

$$\delta(\theta) = [(1/m) \| J(\theta) \|_F^2]^m; \qquad (18)$$

and also [1]:

$$J(\theta)J^T(\theta) = \alpha I; \qquad (19)$$

where, $\alpha \geq 1$.

Here, the considered performance measures are:

$\star$ **C1. Manipulability, [9]** :

$$\delta(\theta) = detJ(\theta)J^T(\theta) \qquad (20)$$

$\star$ **C2. Upper bound on the condition number** :

$$k_2 \leq \| J \|_F^m / \delta^{1/2}(\theta). \qquad (21)$$

$\star$ **C3. Based on isotropy condition** :

- (a) $I_1(\theta) = \delta(\theta) - [(1/m) \| J \|_F^2]^m$;
- (b) $I_2(\theta) = [\| J(\theta)J^T(\theta)) - 2 * I \|_F$.

$\star$ **C4. Proposed upper bound conditioning** :

- (a) $\mu_a \equiv \| J(\theta) \|_s / (l * \Delta t \| \widehat{J}_T(\theta) \|_s)$;
- (b) $\mu_b \equiv \| J(\theta) \|_s / \| \widehat{J}_T(\theta) \|_s$;

where, $\| \widehat{J}_T(\theta) \|_s = [\sum_{j=1}^n \| \widehat{J}_j(\theta)] \|_s$; and $l = max(\dot{\theta})$.

Here, the following Task scenarios are considered:

Task scenarios

I. Manipulator in a initial configuration *far* from singularities is required to move to a final configuration also *far* from singularities, by means of the forward kinematics equation with constant joint velocities.

II. Manipulator in a initial configuration *far* from singularities is required to move to a final configuration *near* to singularities, by means of the forward kinematics equation with constant joint velocities.

III. Manipulator in a initial configuration *far* from singularities is required to move to a final end-effector position (with a configuration *near* to singularities) along a straight line, by means of an inverse kinematics equation with constant end-effector velocities.

IV. Manipulator in a initial configuration *near* to singularities is required to move slightly to a final end-effector position along a straight line, by means of an inverse kinematics equation with constant end-effector velocities.

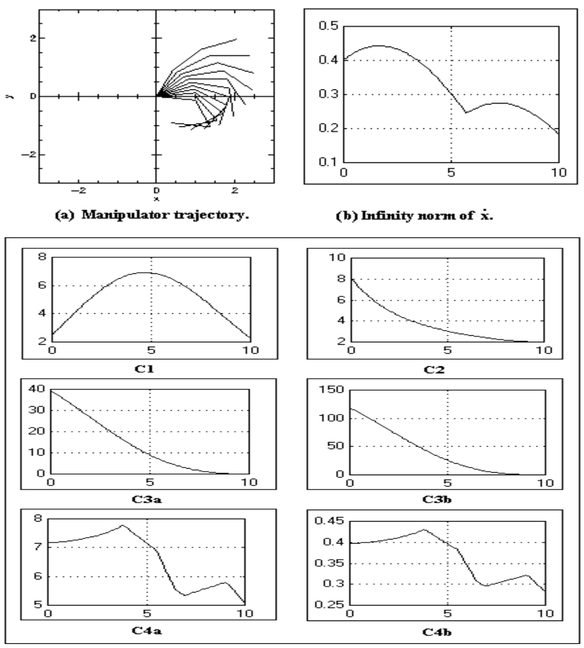
4.1 Cases of study and results

In order to compare several proposed criteria for simplicity a 3 DOF planar redundant manipulator [9] is considered. In this case $Z = I_2$; $W = I_3$, and the defined *standard* norm reduces to the usual norm. Here for convenience the ∞-norm is considered. For all cases the task interval considered was 10 seconds with a step size of 0.1 seconds. For Cases I-III the length of the links is: $l_1 = l_2 = l_3 = 1.0$; whereas, for Case IV $l_1 = 0.60$, $l_2 = 0.85$, $l_3 = 0.20$. The simulation of the Forward and Inverse Kinematics were implemented by means of the Simulink/MATLAB package. For the Cases III and IV the Inverse Kinematics method presented in [8] is utilized.

Case I

-Motions from Forward Kinematics. The initial and final values are specified as follows:

t	θ_1	θ_2	θ_3	x_1	x_2
t_0	0.43	0.35	0.43	2.0558	1.9729
t_f	1.07	1.047	2.096	-	-


(a) Manipulator trajectory. (b) Infinity norm of $\dot{x}$.

(c) Criteria behavior.

Figure 1: Case I.

By feeding the values of the angles at time t to a special purpose graphics package, the entire manipulator motion can be depicted graphically as in Figure 1a. Figure 1b plots the behaviour of the variable $\| \dot{x} \|_\infty$ and Figure 1c. shows the graphs obtained for the different criteria along the joint trajectory.

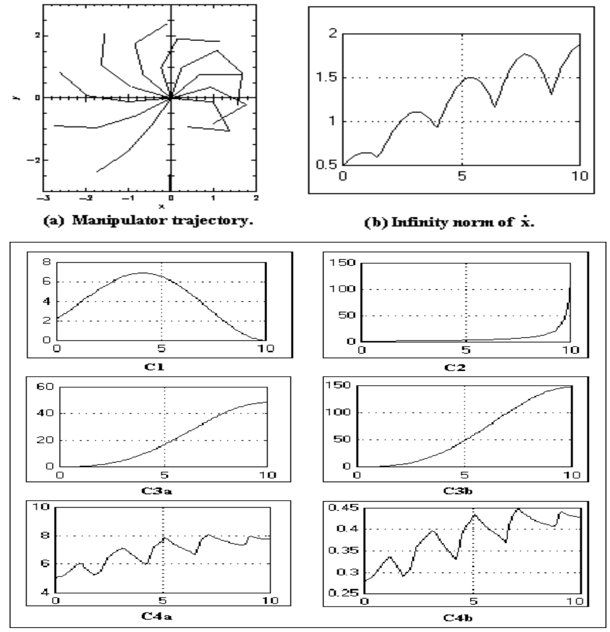
Case II

- Motions from Forward Kinematics

The length of the links are as in the Case I. However, now the initial and final values are specified as follows:

t	θ_1	θ_2	θ_3	x_1	x_2
t_0	1.7	1.047	2.096	0.3846	-0.9218
t_f	-3.1416	0	0.05	-	-

Figure 2a depicts the entire manipulator motion; Figure 2b plots the behaviour of the variable $\| \dot{x} \|_\infty$; and Figure 2c shows the graphs obtained from MATLAB for the different criteria along the joint trajectory.


(a) Manipulator trajectory. (b) Infinity norm of $\dot{x}$.

(c) Criteria behavior.

Figure 2: Case II.

Case III

- Motions from Inverse Kinematics In this case the length links and initial values are the same as in case II. However, now the final end-effector position is specified as follows:

t	θ_1	θ_2	θ_3	x_1	x_2
t_0	1.7	1.047	2.096	0.3846	-0.9218
t_f	-	-	-	-0.0448	-2.990

The final joint configuration obtained is:

θ_1	θ_2	θ_3
3.2375	-0.0735	-0.0954

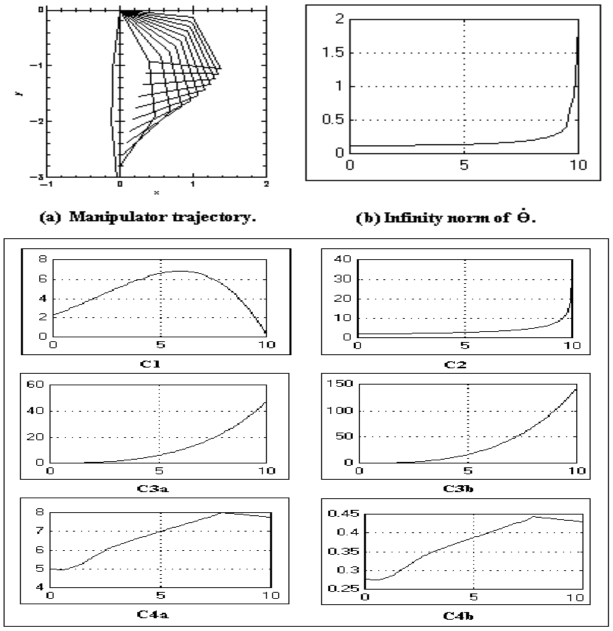
Figure 3a depicts the entire manipulator motion; Figure 3b plots the behaviour of the variable $\| \dot{\theta} \|_\infty$ and Figure 3c shows the graphs obtained for the different criteria along the joint trajectory.

Case IV

- Motions from Inverse Kinematics

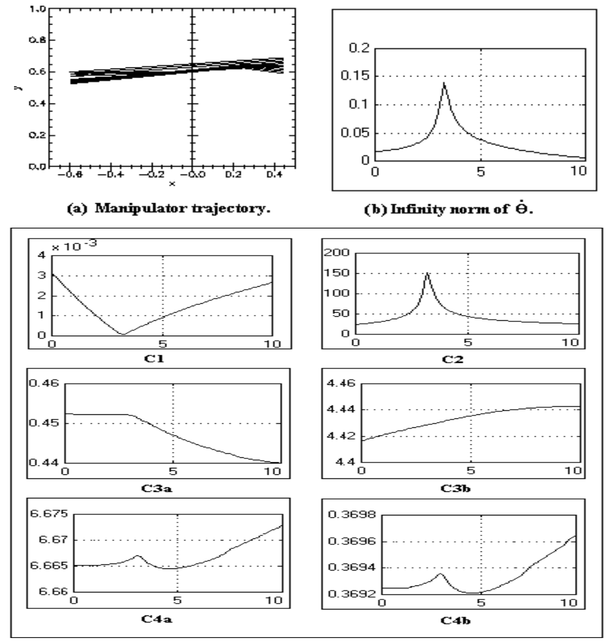
The initial and final values are specified as:

t	θ_1	θ_2	θ_3	x_1	x_2
t_0	-1.5708	3.0543	0.0	0.4460	0.7915
t_f	-	-	-	-0.4460	0.6915



(a) Manipulator trajectory.
(b) Infinity norm of $\dot{\Theta}$.
(c) Criteria behavior.

Figure 3: Case III.



(a) Manipulator trajectory.
(b) Infinity norm of $\dot{\Theta}$.
(c) Criteria behavior.

Figure 4: Case IV with $v = 0$.

For $v = 0$; the Figure 4a shows the entire manipulator motion. The resultant final joint configurations are:

θ_1	θ_2	θ_3
-1.6494	3.135	0.2522

The behaviour of the variable $\| \dot{\theta} \|_\infty$ and the criteria are shown in figures 4b and 4c, respectively.
For the case $v \neq 0$; the Figure 5a depicts the entire manipulator motion. The resultant final joint configurations are:

θ_1	θ_2	θ_3
-0.4750	2.4697	1.4222

Figures 5b and 5c plot the behaviour of the variable $\| \dot{\theta} \|_\infty$ and the criteria, respectively.

4.2 Analysis and Interpretation

From the obtained simulation results the following observations can be made:

- **Case I.** The Figures 1a and 1b, confirm a successful simulation. From the Figure 1c, it can be observed that the criteria C2 to C4b, behave similarly as the manipulator approaches a configuration which is the *most distant* from singularities. Whereas, the criterion C1 yields a maximum around mid task, and an incongruently small value at the end of the task.

- **Case II.** The Figures 2a and 2b, also confirm a successful simulation. From the Figure 3c it can be observed that the criteria C2 to C4b, behave similarly as the manipulator approaches a configuration that is *near* singularities. Whereas, the criterion C1 yields again a maximum around mid task, in this case it does yield an expected very small value at the end of the task.

- **Case III.** The Figure 3a also confirms a successful simulation. From the Figure 3b it can be observed that the norm of the joint velocities becomes very large as the manipulator approaches a singular configuration. From the Figure 3c it can be observed that the criteria C2 to C4b, behave similarly as the manipulator approaches a configuration that is *near* singularities. Whereas, the criterion C1 yields again a maximum around mid task, in this case it does yield an expected very small value at the end of the task.

- **Case IV.** The Figures 4a, 4b and 5a, 5b also confirm a successful simulation.

  For the case in which $v = 0$ the manipulator remains *near* a singular configuration as shown in Figure 4a. In this case the joint velocities become very large as shown in Figure 4b. From the Figure 4c it can be observed that the criteria C1, C2, C4a, and C4b are congruent with the theory. It can be observed that the C1 yields a minimum value that is congruent with the maximum value yielded by C2. Whereas, C3a and C3b yield non-meaningful results.

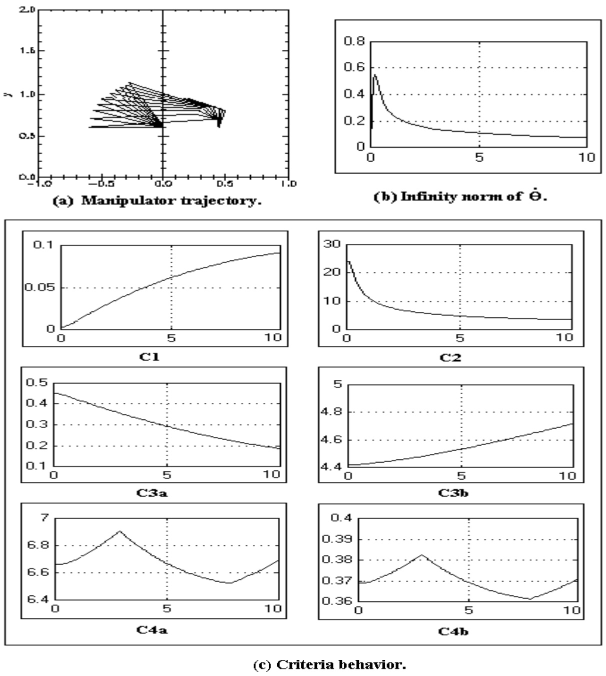


Figure 5: Case IV with $v \neq 0$.

For the case in which $v \neq 0$ the manipulator arranges itself to a configuration *far* from singularities as shown in Figure 5a. In this case, as shown in Figure 5b, although the joint velocities are initially large, they get smaller as the manipulator proceeds to configurations distant to a singular configuration. From Figure 5c it can be observed that the criteria C1, C2, C3a, C4a, and C4b are congruent with the theory. It can be observed that the C1 yields a minimum value that is congruent with the maximum value yielded by C2. In this case C4a, and C4b, reach a maximum value that is not coincident with the maximum yielded by C2; however, the yielded minimum values are congruent with the theory. In this case C3a yields decreasing values as the manipulator moves away from singularities, whereas C3b yields incongruent increasing values.

The results obtained confirm the merit of the proposed criterion. It is worth to mention that unlike criteria C1, and C2, it is useful for general manipulators. Furthermore, unlike criterion it can provide insight of manipulator performance *near* and *far* singularities.

5 Conclusions

In this article a simple index to evaluate overall manipulator motion planning performance is presented. This index is derived by developing a *standard* condition number of the Jacobian matrix an establishing a simple upper bound. Here, the behaviour of the proposed criterion is analyzed/tested on the motions of a planar redundant manipulator (resulting from the forward or inverse kinematic equations) performing several tasks in the entire space. The behaviour and results obtained are compared with the ones obtained using other criteria/measures proposed in the literature. Their results obtained, consistently congruent with the theory, show that the index provides valuable insight of manipulator performance in regions both near and far from singularities. Furthermore, they demonstrate that in spite of its simplicity the proposed index compares favourably with other criteria.

References

[1] Angeles J., Ranjaban F., Patel R. V., *On the Design of the Kinematic Structure of Seven-Axes Redundant Manipulators for Maximum Conditioning*, IEEE Int. Conf. on Robotics and Automation, Nice, France, May 1992.

[2] Klein C. A., Blaho B. E., *Dexterity Measures For The Design and Control of Kinematically Redundant Manipulators*, The Int. Journal of Robotics Research, Vol. 6, No. 2, 1987, pp. 72-83.

[3] Klein C. A., *Use of Redundancy in the Design of Robotic Systems*, The Robotics Research, 2nd. Intl. Symposium, MIT, pp.207-214, 1985.

[4] Kosuge K., Furuta K., *Kinematic and Dynamic Analysis of Robot Arm*, IEEE Int. Conf. on Robotics and Automation, St. Louis, Missouri, Mar. 1985.

[5] Maciejewski A. A., Klein C. A., *The Singular Value Decomposition: Computation and Applications to Robotics*, The International Journal of Robotics Research, Vol. 8, No. 6, Dec. 1989.

[6] Mayorga R. V., E. Díaz de León, "Optimal Upper Bound Conditioning for Manipulator Kinematic Design Optimization", IEEE/RSJ Int. Conf. on Intelligent Robots and Systems, Grenoble, France, Sept. 7-11, 1997.

[7] Mayorga R. V., Milano N., Wong A. K. C., *"A fast Procedure for Manipulator Inverse Kinematics Evaluation and Singularities Prevention*, Journal of Robotics Systems, Vol. 10(1), Feb. 1993.

[8] Mayorga R. V., Wong A. K. C., *A Singularities Prevention Approach for Redundant Robot Manipulators*, IEEE Int. Conf. On Robotics And Automation, Cincinnati, Ohio, May 13-18, 1990.

[9] Yoshikawa T. *Manipulability of Robotic Mechanisms*, International Journal Of Robotics Research, Vol. 4, no. 2, 1985. pp. 3 - 9 .